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HOLE GROWTH IN PLANE VISCOUS CREEP INCLUDING INTERACTION EFFECTS

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Abstract—A basis is presented for modeling the creep rupture process in terms of cavity growth by viscous creep deformation. It is shown that hole interactions must be included in the analysis in order to establish a rupture criterion. A technique introduced by Berg to study the growth of single holes is extended to treat two-hole geometries. The initial displacement rates for the case of two circular holes are calculated and interaction effects are examined.

INTRODUCTION

It has been observed for some time that certain materials become embrittled during service in nuclear reactor environments[1]. Investigators have attributed this effect to the presence of helium bubbles formed in the grain boundaries during irradiation[2]. Decreased strains to rupture then result from the growth and interaction of these bubbles with each other, and with wedge cracks formed at grain boundary triple points by grain boundary sliding[3–5].

In addition to these more critical helium embrittlement phenomena, it has been established that materials generally fail intergranularly at high temperatures, often by the growth and coalescence of voids in the grain boundaries[6]. It has been postulated that these voids are nucleated by grain boundary sliding; and grow either by a vacancy diffusion mechanism, plastic deformation of the grain matrix, or a combination of the two[7].

Whether one is interested in examining the effective weakening of grain boundaries at high temperatures due to the presence of helium bubbles, or in focusing on the problem of creep rupture in general, it is clear that a study of the growth behavior of cavity type defects is important to the understanding of the high temperature deformation and rupture properties of materials. Consequently, there have been a number of investigations aimed an understanding such processions.

As mentioned above, one approach to this problem is to analyze the growth of voids by a vacancy diffusion mechanism. These studies are based on a treatment by Baluffi and Seigle[8], in which they proposed that voids grow, by vacancy absorption, at nuclei lying in grain boundaries transverse to an applied tensile stress. It is shown that there is a thermodynamic driving force for this process, and that a sufficient number of vacancies can be generated in the grain boundaries to supply the growing cavities. Theories of creep rupture based on this model are typified by an analysis made by Hull and Rimmer[9]. They obtained a relation for the spherical void growth rate controlled by vacancy diffusion in the grain boundary, and from it an expression for the rupture time. They also compared this theory to the results of some creep rupture experiments performed on polycrystalline copper samples. Speight and Harris[10] extended this analysis and obtained an expression giving a void growth rate which does not decrease as sharply, with increasing void spacing, as that predicted by the Hull and Rimmer treatment. Recently, Weertman[11] added the effect of volume diffusion and vacancy generation within the grain to the Hull and Rimmer analysis. He also examined the growth of cylindrically shaped cavities by this method. Generally, these theories lead to an expression for the rupture time which is proportional to the cube of the void spacing and the reciprocal of the applied stress.

All of these stress-assisted vacancy-diffusion controlled analyses fail to predict an often observed phenomenon in creep rupture; that is, an apparent relationship between the creep and rupture processes. Experimental results have indicated that the measured steady state creep rates and time to rupture have the same stress and temperature dependence[12–14]. Even the results obtained by Hull and Rimmer[9] for copper illustrate this phenomenon. A nucleation theory combined with the above models might provide an additional stress dependence through the void spacing; but agreement between the resulting net stress dependence and the power law in creep would be fortuitous. Hence, if rupture is a consequence of cavity growth, the above observation leads one to the prospect that this growth is dominated by the creep deformation process itself.

Another approach to the problem of failure resulting from the growth and coalescence of cavities is to consider that they grow by plastic deformation. One of the most familiar such treatments is that presented by McClintock[15] for ductile failure at low temperatures. Using an elastically rigid-perfectly plastic constitutive relation he studied the growth behavior of cylindrical holes under combined uniaxial and radial applied tension. This model predicts finite fracture strains only for non-zero applied radial tension. This is a result of ignoring the interaction between cavities.

Other treatments applying plasticity theory to this problem usually employ a variation principle in conjunction with finite element analysis[16] or some other numerical technique[17]. These treatments usually ignore the interaction between voids. Needleman[18], however, used such methods to study void interactions by treating the deformation behavior of a periodic array of holes.

The constitutive relations for plastic flow applied in the above analyses are inappropriate for a description of creep and creep rupture, since they define a material which yields instanteously above some equivalent stress level. A better equation for high temperature deformation would be one that involves the concept of time dependent flow and predicts finite strain rates which depend upon the applied stress level.

The purpose of this paper is to develop the foundations for a theory of high temperature rupture based on void growth by creep deformation because such an approach inherently includes the connection between the deformation and rupture processes. It is implicitly assumed that the voids exist initially at some finite size, thereby ignoring the complexities of nucleation. In addition, the importance of including void interactions to obtain a rupture criterion without the necessity of an applied biaxial stress state is explored. This is significant for materials which fail in a brittle manner at high temperatures, with little or no necking. The approach presented here is an extension of a technique outlined by Berg[19] for a single hole cases. Application of this method is illustrated for a two-hole geometry and the necessary initial deformation behavior is presented.

2. VISCOUS CREEP CONSTITUTIVE LAW APPLIED TO NON-INTERACTING CAVITIES

Throughout this work, material deformation will be described by the phenomenological equations of viscous creep. In general, these equations state that the rate of deformation is linearly related to the applied stress. This is only rarely a good constitutive law for the creep of metals, but it does apply to glasses and certain polymers. In addition, it contains the essential features of high temperature flow, which allowing for a tractable, analytical solution of some problems. A more realistic power law constitutive relation would prevent the straight-forward treatment of the problems of interest here. However, certain aspects of high temperature failure based on cavity growth by power law creep are considered by Nix *et al.*[20].

Formally, these equations can be written in tensor form as follows:

$$\dot{\epsilon}_{ij} = \frac{\tilde{\sigma}_{ij}}{2\beta}, \tag{1}$$

where $\dot{\epsilon}_{ij}$ are the components of the strain rate tensor, $\tilde{\sigma}_{ij}$ represents the deviatoric part of each component of the stress tensor, and β is the shear viscosity coefficient. The strain rates and displacement rates are related by

$$\dot{\epsilon}_{ij} = \frac{1}{2}(\dot{u}_{i,j} + \dot{u}_{j,i}). \tag{2}$$

The equilibrium equations for viscous creep are precisely the same as those for elasticity. Therefore, the mathematical system that must be solved to analyze viscous creep deformation is essentially that of linear isotropic elasticity. One can examine the growth behavior of a cavity in a viscously deforming medium by obtaining the elasticity solution to the problem, and, by analogy [7], replacing the displacements by displacement rates. Evaluating these rates on the hole boundary gives the initial cavity growth velocity. Once the hole changes shape, the elasticity problem must be re-solved for the new configuration. However, one can obtain insight into the deformation behavior from the initial solution. The complex problem involved in analyzing the continuous deformation of cavities in viscous media will be discussed later.

We now wish to determine whether or not a viable creep rupture model can be evolved when the above formalism is applied to single-hole geometries under uniaxial tension. Such configurations can be used to model arrays of non-interacting cavities. The question is, can such cavities grow in the lateral direction (normal to the tensile axis) so as to meet an imaginary neighbor, leading to rupture? Berg [21] has shown that both elliptical and circular holes contract rapidly and elongate to slits parallel to the tensile axis. We have calculated the initial lateral contraction rate for a circular hole in a two dimensional configuration and for a spherical cavity [22], and found the lateral contraction rate in both cases to be as fast or faster than the rate of Poisson narrowing of the matrix material. These results indicate that such single-hole models cannot be used to treat creep rupture. It is suggested that a failure criterion can be obtained for viscous creep deformation under uniaxial loading only when the interactions between neighboring voids are included. In the subsequent analysis a method is presented which incorporates such interactions in the calculation of cavity growth velocities.

3. CAVITY GROWTH BY VISCOUS CREEP

We now wish to consider the growth of holes by linearly viscous creep deformation, presenting a technique which allows for the inclusion of real interactions between cavities. A class of such problems becomes tractable through the application of complex variable methods to planar model geometries.

As mentioned above, Berg [19, 21] has studied the growth behavior of traction-free single-hole geometries by combining the techniques developed by Muskhelishvili [23] in the mathematical theory of elasticity with the viscous creep analogy. He was able to examine the time-dependent shape change of such cavities by employing the concept of conformal mapping. This procedure will now be extended to treat hole interactions during growth by examining the simplest case of two neighboring traction-free holes in a plate loaded at infinity.

Consider two holes of arbitrary shape in an infinite plane. According to the theory of conformal mapping, a transformation always exists which will map the above geometry into a ring bounded by two concentric circles (Fig. 1). A property of this mapping function is that each boundary of the ring corresponds to one of the hole boundaries. Points in the infinite plane containing two holes, denoted as the z-plane, can be expressed in terms of points in the circular ring in the ξ-plane through a mapping function stated as

$$z = w(\xi). \tag{3}$$

The ring serves as a simple fixed reference geometry in terms of which calculations may be more easily made.

According to the well-known methods of Muskhelishvili, solutions to problems in the plane theory of elasticity are expressed in terms of two analytic complex potential functions $\Phi_1(z)$ and

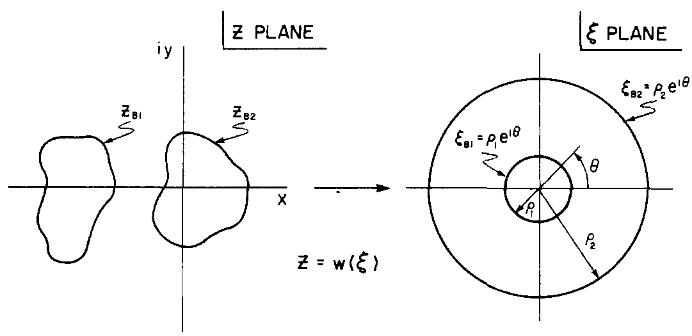

Fig. 1. Two cavities in the z-plane mapped into a ring in the ξ-plane by the function $z = w(\xi)$.

$\psi_1(z)$, where $z = x + iy$ denotes points in the real hole geometry. These functions may be written in terms of the ring coordinates by defining

$$\Phi(\xi) \equiv \Phi_1(z) = \Phi_1(w(\xi)), \tag{4}$$

and

$$\psi(\xi) \equiv \psi_1(z) = \psi_1(w(\xi)). \tag{5}$$

The traction-free boundary condition may be expressed in terms of these functions as

$$\Phi(\xi_B) + w(\xi_B)\frac{\overline{\Phi'(\xi_B)}}{\overline{w'(\xi_B)}} + \overline{\psi(\xi_B)} = C_B, \tag{6}$$

where $\xi_B = \rho_B\, e^{i\theta}$ denotes points on one of the ring boundaries (Fig. 1), the bars denote complex conjugates and C_B is a constant. Employing the viscous creep analogy, the displacement rates on the hole boundary are given by

$$2\beta(\dot{u} + i\dot{v})|_B = \Phi(\xi_B) - w(\xi_B)\frac{\overline{\Phi'(\xi_B)}}{\overline{w'(\xi_B)}} - \overline{\psi(\xi_B)}, \tag{7}$$

where $\dot{u}$ and $\dot{v}$ are the displacement rates in the x and y directions respectively. Combining these equations results in a simple expression for the displacement rates:

$$2\beta\,(\dot{u} + i\dot{v})|_B = 2\Phi(\xi_B) - C_B. \tag{8}$$

This equation gives the displacement rates of points on the hole boundaries in the z-plane in terms of points on the fixed ring boundaries (reference geometry). After a time increment, Δt, the displacements of these points are given by,

$$2\beta\,(u + iv)|_B = [2\Phi(\xi_B) - C_B]\,\Delta t. \tag{9}$$

If initially, at time t_0, the points on the hole boundaries are denoted by $z_B^{(0)} = w^{(0)}(\xi_B)$, then at time $t_1 = t_0 + \Delta t$ the points on the boundaries are

$$z_B^{(1)} = z_B^{(0)} + (u + iv)|_B, \tag{10}$$

or, by eqn (9).

$$z_B^{(1)} = w^{(0)}(\xi_B) + [2\Phi(\xi_B) - C_B]\frac{\Delta t}{2\beta}. \tag{11}$$

Hence the new mapping function is given by

$$z_B^{(1)} = w^{(1)}(\xi_B) = w^{(0)}(\xi_B) + [2\Phi(\xi_B) - C_B]\frac{\Delta t}{2\beta}. \tag{12}$$

The function $w^{(1)}(\xi_B)$ relates the deformed boundaries in the z-plane to the fixed ring. This new mapping function can be used as an aid to solving the elasticity problem pertaining to the new hole boundaries, $z_B^{(1)}$. In this way the growth behavior of the original cavities, including interaction effects, can be examined incrementally.

A general mapping function which transforms an infinite plane containing two holes of arbitrary shape into a ring would have the form

$$z = w(\xi) = \frac{R\xi}{1 - a\xi} + \sum_{-\infty}^{\infty} k_n \xi^n, \tag{13}$$

i.e. a singular part plus a holomorphic part, where a corresponds to the point $z = \infty$. Regardless of the nature of this function, the elasticity-viscous creep solution for general loading at infinity can be obtained, in principle, using the approach discussed above. This will give the initial hole growth velocity, and the initial shape change after a time increment. A particular two-hole configuration can be mapped into a ring by truncating the series and adjusting the constants appearing in eqn (13) such that

$$w(\xi) = \frac{R\xi}{1 - a\xi} + \sum_{n=-N}^{M} k_n \xi^n. \tag{14}$$

If the mapping function which takes the original two holes into a ring, at $t = t_0$, is denoted by

$$z_B^{(0)} = w_B^{(0)}(\xi) = \frac{R^{(0)}\xi_B}{1 - a\xi_B} + \sum_{n=-N}^{M} k_n^{(0)} \xi_B^n, \tag{15}$$

then, after the increment Δt, the new function will be such that

$$w^{(1)}(\xi_B) = w^{(0)}(\xi_B) + (\dot{u} + i\dot{v})\Delta t, \tag{16}$$

where the displacement rates are known as a function of the fixed ring coordinates. Berg has shown for the single hole case that the potential function, $\Phi(\xi)$, from which the displacement rates are obtained (eqn (8)) is truncated to the same number of terms as the mapping function. In the present case, this would imply that $\Phi(\xi)$ would have the form [23]

$$\Phi(\xi_B) = \Gamma R^{(0)} \frac{\xi_B}{1 - a\xi_B} + \sum_{-N}^{M} a_n \xi_B^n.$$

Then the new mapping function would be given by

$$w^{(1)}(\xi_B) = R^{(1)} \frac{\xi_B}{1 - a\xi_B} + \sum_{-N}^{M} k_n^{(1)} \xi_B^n. \tag{17}$$

Therefore, if the shapes of the two holes are described within one family (i.e. $w(\xi)$ is truncated with a certain number of terms), they will change to a shape within the same family under the loading at infinity. The determination of the continuous hole growth behavior involves obtaining the time dependent mapping function

$$w(\xi, t) = R(t) \frac{\xi}{1 - a\xi} + \sum_{n=-N}^{M} k_n(t) \xi^n. \tag{18}$$

This problem is outside the scope of the present work, and will be examined in later studies.

The simplest case of two equal-sized circular holes presents an exception to the above form (eqn 14). The initial mapping for this configuration involves only the singular part, i.e.

$$w^{(0)}(\xi) = \frac{\xi}{1 - a\xi}. \tag{19}$$

Under general loading the holes will deform such that the new mapping function will take the general form of eqn (14). Therefore, the initial solution must be obtained in order to determine the number of terms in the time-dependent mapping function. It is also an essential first step in an incremental solution. For these reasons, and to illustrate the effect of including hole interactions, the initial displacement rates for the case of two equal-sized circular cavities in a linearly viscous plate loaded at infinity will be presented here.

Although the problem of two circular holes in a plate under uniaxial tension has been treated using the Airy stress function approach [24–28], the displacements were not determined by these workers because they focused mainly on stress concentrations at certain points in the plate. To obtain the displacements, one would have to apply complex integration techniques [29] to the infinite series solutions involved in these earlier analyses. The approach outlined by Muskhelishvili[23] allows for the direct determination of the displacements of the hole boundaries. By applying this method to the two-hole configuration shown in Fig. 2 the following expression for the displacement rates is obtained (see Appendix and[30]):

$$2\beta(\dot{u}+i\dot{v})|_{Q_1} = 2\left[-\frac{a\xi_1}{1-a\xi_1}\sum_{n=-\infty}^{\infty}(B_n+B_{-n})(a\bar{\xi}_1)^n - \sum_{n=-\infty}^{\infty}B_n(a\bar{\xi}_1)^n\right] - \frac{T}{2a}\left[\frac{a\xi_1}{1-a\xi_1}+\frac{2a\bar{\xi}_1}{1-a\bar{\xi}_1}-\frac{1}{2}\right] \equiv g(\xi_1) \quad (20)$$

where Q_1 and the subscript 1 refer to one of the holes, $\xi_1 = \rho_1 e^{i\theta}$ refers to points on the ring boundary corresponding to that hole (see Fig. 2), and T is the uniaxial tension applied normal to the common axis of the holes. The constants B_n, are related to the series coefficients in the expansion of the potential function $\psi(\xi) = \sum_{-\infty}^{\infty} b_n\xi^n + \Gamma(\xi/1-a\xi)$ such that $B_n = (b_n/a^n)$. Although the series in eqn (20) are written with infinite limits, only twelve terms are required for convergence, The displacement rates of the other hole can be obtained from symmetry.

The displacement rates of the hole boundary described by eqn (20) are expressed as some function of points on the boundary of the reference ring. The displacements after a time increment, Δt, are given by

$$2\beta(u+iv) = g(\xi_1)\Delta t. \quad (21)$$

In Fig. 3 the initial shape change of the hole boundaries is shown as calculated from eqns (20) and (21). The time increment was chosen for illustrative purposes, but corresponds to an axial strain of about 10% for typical laboratory applied stress and steady state strain rate values. One important aspect to be noted from the results shown in Fig. 3 is the effect of hole interactions. It was indicated earlier that single holes contract laterally as fast or faster than the Poisson narrowing of the matrix material under uniaxial loading, and hence that void coalescence could not be simulated in this case. Table 1 shows the ratio of the initial lateral hole contraction rate to the Poisson narrowing rate for the two-hole problem considered here. The results show that the initial hole contraction rate is slower when interactions are included. This admits the possibility of developing a rupture criterion, since the interaction is expected to become stronger as the cavities grow.

SUMMARY AND CONCLUSIONS

Creep rupture and its enhancement by irradiation often results from the presence and growth to coalescence of cavities and bubbles in the grain boundaries of polycrystalline materials. Since a relationship between the steady-state deformation and rupture processes has been experimentally observed, it is conceivable that these cavity-type defects grow by creep deformation.

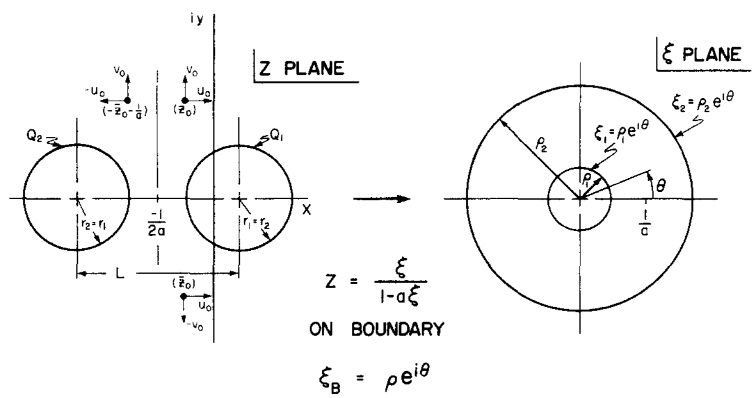

Fig. 2. Two circular holes in the z-plane mapped into a ring in the ξ-plane by the function $z = \xi/(1-a\xi)$.

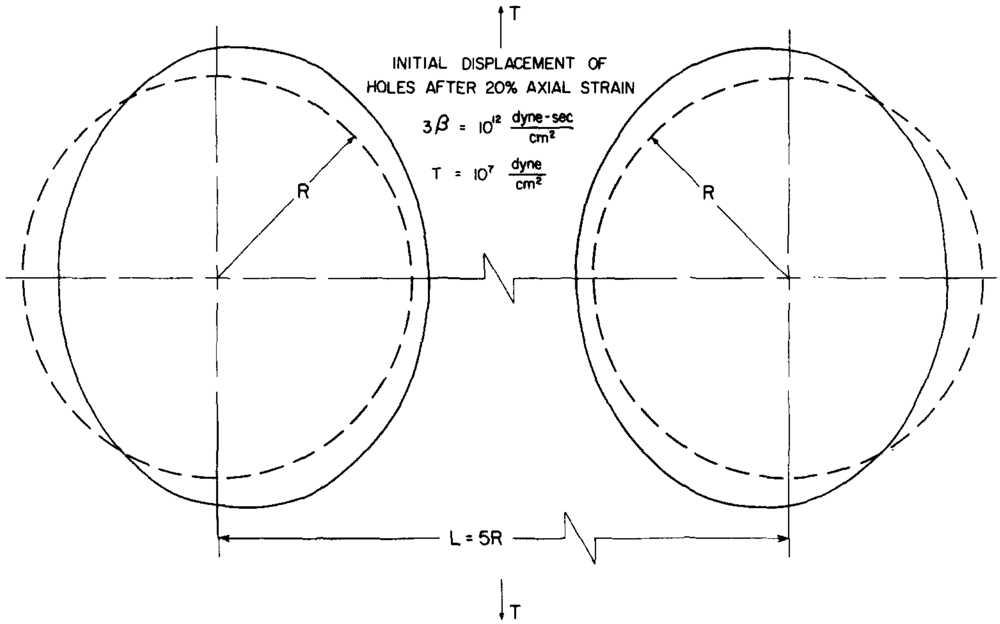

Fig. 3. Initial displacements of hole boundaries during hole growth under uniaxial tension applied at infinity.

Table 1. Ratio of lateral hole contraction rate to Poisson contraction

Hole Spacing, L (in terms of hole radius, R)	Contraction Rate Ratio
3 x R	0.89
4 x R	0.91
5 x R	0.98
10 x R	0.98
∞ (single hole)	1.00

In the present analysis we have considered the growth of cavities by plastic deformation using a viscous creep constitutive relation, in order to develop the basis for a creep rupture model. We have concluded from preliminary calculations that a failure criterion cannot be evolved for non-interacting holes growing under uniaxial tension. The problem of hole interactions has been approached by extending a complex variable technique developed by Berg[19] for single holes to two-hole configurations. In this technique the deforming cavities are continuously mapped into a simple reference geometry. The results thus far far have indicated that the inclusion of hole interactions admits the possibility of developing a rupture criterion. The initial displacements of two circular holes have been explicitly determined for the first time, and the procedure for examining their continuous growth has been outlined. This problem has been left as the subject of later study.

The approach presented here has other important applications. For example, the problem of surface blistering resulting from the growth of non-equilibrium gas bubbles formed near the surface of irradiated materials can be examined, since a bubble deforming beneath a planar surface can be readily mapped in a simple geometry.

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APPENDIX

Initial displacement rates for two circular holes

The infinite plane containing two equal-sized circular holes is mapped into a ring (Fig. 2) by the transformation

$$z = w(\xi) = \frac{\xi}{1 - a\xi}. \tag{A1}$$

Defining the functions $\Phi(\xi)$ and $\psi(\xi)$, Muskhelishvili[23] shows that the condition of zero tractions on the hole boundaries may be written as

$$\Phi(\xi_B) + w(\xi_B)\overline{\frac{\Phi'(\xi_B)}{w'(\xi_B)}} + \overline{\psi(\xi_B)} = C_B \tag{A2}$$

for each hole boundary expressed in terms of points on the ring boundary, ξ_B. For viscous creep deformation the displacement rates can be written as

$$2\beta(\dot{u} + i\dot{v}) = = \Phi(\xi) - w(\xi)\overline{\frac{\Phi'(\xi)}{w'(\xi)}} - \overline{\psi(\xi)}, \tag{A3}$$

and the stresses as

$$\sigma_{xx} + \sigma_{yy} = 2\left[\frac{\Phi'(\xi)}{w'(\xi)} + \overline{\frac{\Phi'(\xi)}{w'(\xi)}}\right], \tag{A4}$$

and

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\overline{w(\xi)}\frac{\Phi''(\xi)}{(w'(\xi))^2} - \frac{\Phi'(\xi)w''(\xi)}{(w'(\xi))^3} + \frac{\psi'(\xi)}{w'(\xi)}\right].$$

Further, it can be shown[23] that the functions $\Phi(\xi)$ and $\psi(\xi)$ have the form

$$\Phi(\xi) = \frac{T}{4}\frac{\xi}{1 - a\xi} + \sum_{-\infty}^{\infty} a_n\xi^n,$$

and

$$\psi(\xi) = \frac{T}{2}\frac{\xi}{1 - a\xi} + \sum_{-\infty}^{\infty} b_n\xi^n, \tag{A5}$$

where T is the uniaxial tension applied in the y-direction (Fig. 2). Noting the symmetry about the x-axis and the line $x = -(1/2a)$, and defining

$$A_n = \frac{a_n}{a^n}$$

and

$$B_n = \frac{b_n}{a^n}, \tag{A6}$$

the following symmetry relations can be obtained[30]:

$$\begin{aligned} A_k &= \bar{A}_k \\ A_{-k} &= -A_k \\ B_k &= \bar{B}_k \\ B_k + B_{-k} &= (k+1)A_{k+1} - 2kA_k + (k-1)A_{k-1}. \end{aligned} \tag{A7}$$

Substituting (A5)–(A7) into (A2) gives

$$B_0 = A_1,$$
$$A_1 = \frac{aT\rho_1^2}{4(1 - a^2\rho_1^2)},$$
$$B_{-1} = -2A_1,$$

and the following two recursion formulae:

$$[(a\rho_1)^k + k(a\rho_1)^{-(k-2)}]A_k - [(a\rho_1)^k + 2(k-1)(a\rho_1)^{-(k-2)}]A_{k-1} + (k-2)(a\rho_1)^{-(k-2)}A_{k-2} + (a\rho_1)^{-k}B_{-k} - (a\rho_1)^{-(k-2)}B_{-k+1} = 0,$$

and

$$[(a\rho_1)^{-k} + k(a\rho_1)^{k-2}]A_k - [(a\rho_1)^k + 2(k-1)(a\rho_1)^{k-2}]A_{k-1} + (k-2)(a\rho_1)^{k-2}A_{k-2} + (a\rho_1)^k B_{-k} - (a\rho_1)^{k-2}B_{-k+1} = \frac{T}{2a}[(a\rho_1)^k - (a\rho_1)^{k-2}],$$
$$k \geq 2.$$

From these relations, one can obtain the potential functions, $\Phi(\xi)$ and $\psi(\xi)$, and hence the displacement rates[30].